Convective instabilities of synchronization manifolds in spatially extended systems

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We study the stability properties of anticipating synchronization in an open chain of unidirectionally coupled identical chaotic oscillators. Despite being absolutely stable, the synchronization manifold is unstable to propagating perturbations. We analyze and characterize such instabilities drawing a qualitative and quantitative comparison with the convective instabilities typical of spatially extended systems.

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Synchronization of coupled chaotic systems has been the object of intensive studies over the past years [1]. Basically all relevant questions have been investigated and clarified in the context of low-dimensional systems, including the subtle issues connected to the stability of the synchronization manifold that may depend on the transversal Lyapunov exponent [2] or the Lyapunov function [3].

Much less is known about synchronization properties of high-dimensional or, more specifically, extended systems. In this latter context, evidences of synchronization phenomena have been given in large populations of coupled chaotic units and neural networks [4], globally or locally coupled map lattices [5], and in space-extended systems [6].

Among the few general features that have been established, one finds that linear stability analysis may even fail altogether to predict the stability property of the synchronous state [7].

In this paper we focus our interest on the recently discovered anticipating synchronization [8] in chains of unidirectionally coupled oscillators. Here, the (short) time delay τ in the mutual coupling makes the trajectories to converge towards an absolutely stable anticipating synchronization manifold (ASM) wherein the state of the response system anticipates that of the driver by the same amount of time τ. Since absolute stability remains as such independent of the chain length, one is tempted to conclude that arbitrarily long anticipation times may be generated. The weirdness of the seeming lack of causality together with the potential application of this phenomenon in real-time forecasting has suggested us to investigate more in detail the stability of the synchronized regime. As a result we find that the ASM absolute stability is accompanied by a convective instability which undermine the stability of the synchronous regime in long chains.

In order to address such a problem, let us consider an open chain of N unidirectionally coupled identical Rössler oscillators [9], given by

$$\dot{\mathbf{r}}_i = \mathbf{f}(\mathbf{r}_i) + \varepsilon(1-\delta_{1i})[\mathbf{r}_{i-1} - \mathbf{r}_i(t-\tau)], \quad (1)$$

where the dots denote temporal derivatives, $\mathbf{r}_i \equiv (x_i, y_i, z_i)$ is the vector field of the ith driven oscillator $(i=1,\ldots,N)$, ε is the coupling strength, τ is the delay time in the coupling factor, δ_{ij} is the Kronecker δ function, and $\mathbf{f}(\mathbf{r})$ is a vector field,

$$\mathbf{f}(\mathbf{r}) = [-y-z, x+ay, b+z(x-c)], \quad (2)$$

responsible for generating the locally chaotic dynamics. In the following, we set $a=0.15, b=0.2, c=10, N=100$ and we study the evolution of system (1) upon varying τ and ε, starting from a set of random initial conditions $\mathbf{r}_i(0)$ covering all the interval $[0,-\tau]$ for each oscillator. All reported simulations have been performed by implementing a fourth order Runge-Kutta integration scheme with free boundary conditions.

In order to carry on the linear stability analysis, it is convenient to pass from the $\{\mathbf{r}_i(t)\}$ to the $[\mathbf{r}_1(t), \Delta\mathbf{r}_i \equiv \{\mathbf{r}_{i-1}(t) - \mathbf{r}_i(t-\tau)\}]$ representation (with $i>1$). In fact, the synchronized state is characterized by $\Delta\mathbf{r}_i=0$. Linearization of the equations for $\mathbf{r}_1$ accounts simply for the Lyapunov exponents of the single Rössler oscillator. The dynamics of an infinitesimal perturbation $\rho_i=(u_i, v_i, w_i)$ of the differences $\Delta\mathbf{r}_i$ is instead described by

$$\dot{u}_i = -v_i - w_i + \varepsilon(1-\delta_{1i})u_{i-1} - \varepsilon u_i(t-\tau),$$
$$\dot{v}_i = u_i + az_i + \varepsilon(1-\delta_{1i})v_{i-1} - \varepsilon v_i(t-\tau), \quad (3)$$
$$\dot{w}_i = (x_i - c)w_i + z_i u_i + \varepsilon(1-\delta_{1i})w_{i-1} - \varepsilon w_i(t-\tau).$$

The growth rates of $\rho_i (i \geqslant 2)$ define the so called *transversal* Lyapunov exponents, insofar as they give information on the evolution of perturbations transversally to the ASM (represented by the fixed point $\Delta\mathbf{r}=0$), and the negativity of the maximum of such exponents is a necessary condition for absolute stability of such a manifold. It is important to notice that while the dynamical law for ρ_2 is self-contained, the evolution of all other perturbations can be determined only as a cascade process. A necessary condition for the synchronized regime to be stable is that the growth rate

$$\lambda_\perp = \lim_{T\to\infty} \frac{1}{T} \ln \frac{\|\rho_2(T)\|}{\|\rho_2(0)\|}$$

of ρ_2 is negative. Since the Lyapunov exponent $\lambda_\perp$ is a self-averaging quantity, it is sufficient to evolve a single randomly chosen initial condition $[\rho_1(t=0)$ and the set $\{\rho_2(t),$

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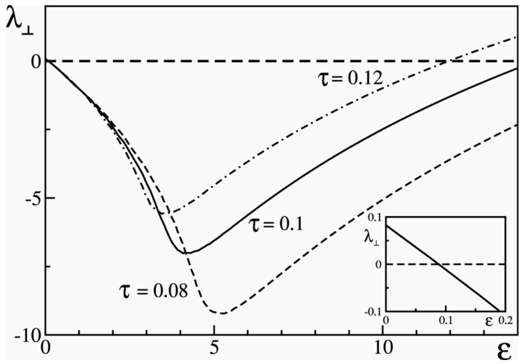

FIG. 1. Transversal Lyapunov exponent $\lambda_\perp$ (see text for definition) computed from system (3) vs the coupling strength ε for $\tau=0.12$ (dot-dashed line), $\tau=0.1$ (solid line), and $\tau=0.08$ (dashed line). The three curves start from $\lambda_0 \simeq 0.0826$, corresponding to the maximum (positive) Lyapunov exponent for the Rössler oscillator.

$-\tau \leqslant t \leqslant 0\}]$ for a time T long enough. In Fig. 1, we have plotted the values of $\lambda_\perp$ vs the coupling strength, for different choices of τ, for $T=1000$ and adopting as norm $\|\cdot\|$ the maximum of the absolute values of the three components of ρ. Consistently with what observed in Ref. [8], $\lambda_\perp$ is negative for a suitable parameter range, indicating that the ASM is there absolutely stable. Notice that for all choices of τ, $\lambda_\perp$ at zero coupling is positive and equal to the maximum Lyapunov exponent of the single Rössler system, $\lambda_0=0.0826$ (for the chosen parameter values). For the relatively small τ values considered in Fig. 1, one can verify that $\lambda_\perp$ approximately scales as $\lambda_\perp \approx (1/\tau)L(\varepsilon\tau)$. The scaling $1/\tau$ is what is well known to occur in delayed systems in the limit of very long delay times. It is therefore remarkable here to observe the very same scaling properties at already small values of τ. As for the dependence on $\varepsilon\tau$, this would imply a maximal stability for $\varepsilon \simeq 1/\tau$. However, from Fig. 2 of the first of Refs. [8], one clearly sees that stability is completely lost for $\tau \gtrsim 0.8$, thus implying that such a scaling property holds only for τ values small enough.

On the basis of the results reported in Fig. 1, one is tempted to conclude that arbitrarily long anticipation times can be obtained by just coupling a sufficiently large number N of oscillators. Since the ith oscillator anticipates its driver by a time τ, its dynamics is expected to collapse onto a manifold wherein $\mathbf{r}_i(t)=\mathbf{r}_1[t+(i-1)\tau]$. In fact, this would be possible only if absolute stability were a sufficient condition for the settings of such a manifold. Figure 2 indeed shows that this is not the case. System (1) is evolved from random initial condition for $N=100, \tau=0.1$, and $\varepsilon=4.1$ (from the solid curve in Fig. 1, one can clearly see that the corresponding $\lambda_\perp$ is negative) up to the time at which the ASM is reached. At this point a zero average δ-correlated Gaussian noise $D\xi(t)$ of small amplitude $D=0.005$ is added to the variable y_1. The deviations from the ASM are thereby monitored by evaluating $\delta x_i = |x_1(t) - x_i(t-(i-1)\tau)|$. From Fig. 2 it is clear that the trajectory abandons the absolutely stable ASM manifold $\delta x_i=0$ as a result of the applied perturbation although the deviations in the fifth oscillator are still quite small. In fact, it is crucial to add that the asymptotic (in time) size of the deviations tend to grow exponentially with i.

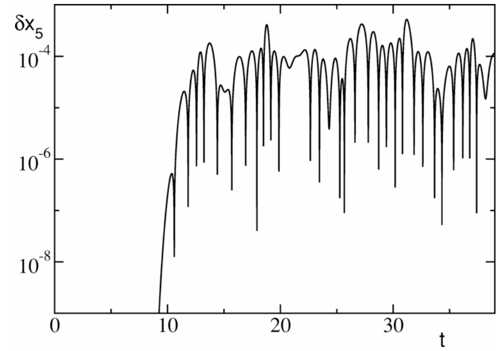

FIG. 2. Temporal evolution of δx_5 for $\tau=0.1$ and $\varepsilon=4.1$. The trajectory starts from random initial conditions and is subjected to a zero average δ-correlated Gaussian noise perturbation $D\xi(t)$ ($D=0.005$) added to the variable y_1.

In order to clarify the whole process, we prefer to investigate the response of the system to a δ-like perturbation. More precisely, we have let the system (1) evolve from random initial condition at $t=0$ (with $\tau=0.1$ and $\varepsilon=4.1$) until it reaches (within numerical accuracy) the ASM. Then, evolution is restarted after perturbing x_1 by a small amount η, while all other variables are left unchanged. Convergence back to the ASM is studied by monitoring the single step anticipation error $\sigma_i^2 = \langle [x_i(t-\tau) - x_{i-1}(t)]^2 \rangle$, where angular brackets denote an average over an ensemble of independent choices of the initial conditions.

In the limit of small perturbations, instead of following two separate trajectories, it is sufficient to let a perturbation evolve in tangent space: in this limit $\sigma_i^2 = \langle u_i^2 \rangle$. The curves corresponding to different oscillators that are plotted in Fig. 3 clearly indicate that the deviation from the ASM initially grows but eventually converges to 0 thus confirming its absolute stability. On the other hand, oscillators labeled by larger i values are characterized by higher peaks. Figure 3 also demonstrates that the behavior of the system is basically insensitive on whether calculations are performed in the normal or in the tangent space.

This phenomenon is very much reminiscent of convective instabilities in spatially extended systems where a localized

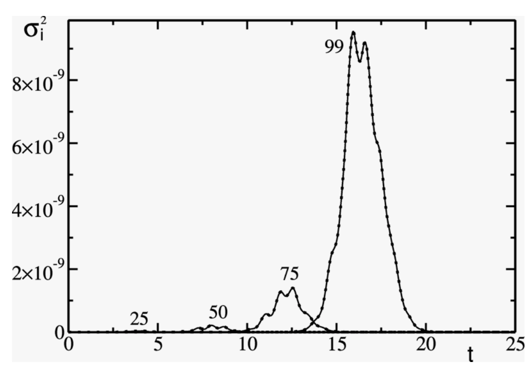

FIG. 3. Time evolution of the ensemble averaged differences $\sigma_i^2 = \langle u_i^2 \rangle$ for $i=25, 50, 75$, and 99 (the corresponding numbers are on the top of each curve). Data is obtained from an ensemble average of 10 000 perturbations, for $\tau=0.1$, $\varepsilon=4.1$, and $\eta=5\times10^{-3}$. The solid (dotted) lines refer to phase (tangent) space.

perturbation dies if observed where it has been generated while it appears to grow in suitably moving frames. The analogy relies on the interpretation of the integer i labeling the oscillators as a space variable, but an exact mapping with convective phenomena is hindered by the additional presence of the "delayed" interactions which make the problem conceptually more complex.

One can, nevertheless, test whether the evolution of an initially localized perturbation follows the same scaling behavior as in spatially extended systems. In the context of one-dimensional lattices, the convective Lyapunov exponent is defined as [10]

$$\Lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta(i = vt, t)|}{|\delta(0,0)|}, \tag{4}$$

where $\delta(i,t)$ denotes the perturbation amplitude in site i at time t and is initially localized in a finite region around the origin. This is equivalent to stating that

$$\delta(i,t) \simeq \exp(\Lambda(v)t) = \exp\left(\frac{\Lambda(v)}{v} i\right) \tag{5}$$

for both $|i|$ and t are large enough.

From a numerical point of view, $\Lambda(v)$ can be accurately estimated by comparing the perturbation amplitude at two different space-time positions $P_1 \equiv (i_1, t_1)$, $P_2 \equiv (i_2, t_2)$,

$$\Lambda(v) = \frac{v}{i_2 - i_1} \ln \frac{|\delta(i_2, t_2)|}{|\delta(i_1, t_1)|}, \tag{6}$$

where $v = i_1/t_1 = i_2/t_2$. In fact, provided that both P_1 and P_2 are far enough from the origin, multiplicative finite-size corrections affect δ in the same way and thus disappear when the ratio is taken in Eq. (6).

The results reported in Fig. 4 confirm that the behavior of perturbations in the context of Rössler oscillators with delayed coupling is analogous to that of convectively unstable systems. Indeed, the three curves obtained by comparing the following pairs of oscillators, (80,60), (60,40), and (40,20) almost overlap, thus suggesting that the convective spectrum $\Lambda(v)$ is a well defined quantity in this context too. Next, the very existence of a positive maximum of $\Lambda(v)$ implies that perturbations traveling with a velocity v in between the two zeros of $\Lambda(v)$ (approximately equal to 5 and 8) are indeed amplified. Furthermore, the maximum rate, approximately equal to 0.203, is larger than the positive Lyapunov exponent of the single oscillator λ_0, indicating that such convective instability is even stronger than local instability. The value of the maximum convective exponent can be independently checked by monitoring the values of the maxima σ_M of each σ_i versus their occurrence time. The best fit reported in the inset of Fig. 4 corresponds to a growth rate of 0.202, in good agreement with the maximum of the convective spectrum.

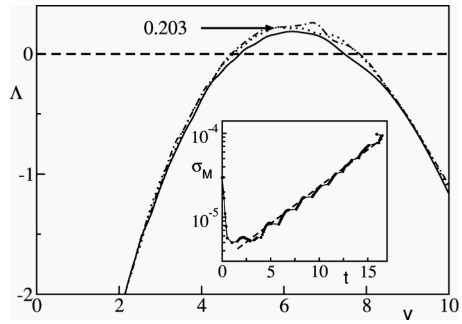

FIG. 4. Convective Lyapunov exponent $\Lambda(v)$ vs propagation velocity v, computed by comparing the perturbation in different pairs of oscillators according to Eq. (6). Dot-dashed, dotted, and solid lines correspond to the pairs (80,60), (60,40), and (40,20), respectively. The maximum value of the exponent is marked by an arrow. Inset: maximum value of σ_i vs its time of occurrence. The exponential best fit yields an exponent 0.202, in agreement with the maximal convective Lyapunov exponent.

Since the scaling behavior sets in only above $t \gtrsim 2$, and the maximal convective instability corresponds to a velocity $v \approx 6$, this means that this effect originates only for chain lengths larger than about ten oscillators.

In conclusion, we have shown that convective instability prevents the occurrence of anticipating synchronization over arbitrarily long times in a chain of unidirectionally coupled identical chaotic oscillators, when even a small amount of noise is present.

This evidence indicates that absolute stability of the synchronization manifold is only a necessary condition for the robustness of synchronization properties in coupled spatially extended systems, and other types of space-time instabilities have to be taken into account.

A general consequence is that necessary and sufficient conditions for the stability of synchronization properties in spatially extended systems strictly depend on the space-extended nature of the dynamics and need to be assessed by taking into account additional sources of instability, such as convective growth of perturbations in moving frames.

A further consequence concerns the possibility of implementing anticipating synchronization as a strategy for real-time forecasting of future states of a given dynamics. Such a possibility needs to be reconsidered by a careful investigation of the space-time instabilities that might be suffered by the synchronized dynamics when noisy perturbations are taken into account.

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